# Fixed point Theorems for Expansive mappings in G-metric Spaces

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**Abstract:** In this Paper, we define the expansive mappings in the setting of G-metric space; also several fixed point theorems for a class of expansive mappings defined on a complete G-metric space are studied.

**Mathematics Subject Classification**: Primary 47H10, Secondary 46H20

**Keywords:** Metric space, generalized metric space, G-metric space, 2-metric space, expansive mapping.

**Introduction:** In 2005, a new structure of generalized metric spaces was introduced by Zead Mustafa and Brailey Sims as appropriate notion of generalized metric space called G-metric spaces see [3] as follows.

**Definition 1**. [(3)] Let $X$ be a nonempty set, and let $G: X \times X \times X \to R^+$, be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (Symmetry in all three variables), and

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function $G$ is called generalized metric, or, more specially G-metric on $X$, and the pair $(X, G)$ is called a G-metric space. (Throughout this paper we denote $R^+$ the set of all positive real numbers and $N$ the set of all natural numbers).

**Definition 2.** Let $(X, G)$ be a G-metric space, let $\{x_n\}$ be a sequence of points of $X$, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one say that the sequence $\{x_n\}$ is G-convergent to $x$.

Thus, that if $x_n \to 0$ in a G-metric space $(X, G)$, then for any $\varepsilon > 0$, there exists $n \in N$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

**Proposition 1.** ([3]) Let $(X, G)$ be a G-metric space. Then the following are equivalent.

$(x_n)$ is $G$-convergent to $x$.

$G(x_n, x_n, x) \to 0$, as $n \to \infty$.

$G(x_n, x, x) \to 0$, as $n \to \infty$.

$G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

**Definition 3.** ([3]) Let $(X, G)$ be a G-metric space. A sequence $(x_n)$ is called G-Cauchy if given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

**Proposition 2.** ([3]) If $(X, G)$ be a G-metric space, then the following are equivalent.

1. The sequence $(x_n)$ is G-Cauchy.

2. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

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**Definition 4.** ([3]) Let $(X,G)$ and $(X',G')$ be two G-metric spaces, and let $f:(X,G)\to(X',G')$ be a function. Then $f$ is said to be G-continuous at a point $a\in X$ if and only if given $\varepsilon>0$, there exists $\delta>0$ such that $x,y\in X$ ; and $G(a,x,y)<\delta$ implies $G'(f(a),f(x),f(y))<\varepsilon$ .A function $f$ is G-continuous on $X$ if and only if it is G-continuous at all $a\in X$ .

**Proposition 3.** ([3]) Let $(X,G)$ and $(X',G')$ be two G-metric spaces. Then a function $f:X\to X'$ is G-continuous at $x\in X$ if and only if it is G-sequentially continuous at $x$; that is, whenever $(x_n)$ is G-convergent to $x$ we have $(f(x_n))$ is G-convergent to $f(x)$.

**Proposition 4.** ([3]) Let $(X,G)$ be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.

**Definition 5.** ([3]) A G-metric space $(X,G)$ is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in $(X,G)$ is G-convergent in $(X,G)$.

**Definition 6.** ([3]) A G-metric space $(X,G)$ is called symmetric G-metric space if $G(x,y,y)=G(y,x,x)$ for all $x,y\in X$ .

The following fixed point theorem for a contractive mapping on G-metric space has proved in [2].

**Theorem 1.1**. [2] Let $(X,G)$ be a complete G-metric space and $T:X\to X$ be a mapping satisfies the following condition for all $x,y,z\in X$

$$G(Tx,Ty,Tz)\le kG(x,y,z) \quad (1.1)$$

Where $k\in[0,1]$. Then $T$ has a unique fixed point.

**Theorem 1.2**. ([2]) Let $(X,G)$ be a complete G-metric space and $T:X\to X$ be a mapping satisfies the following condition for all $x,y\in X$

$$G(Tx,Ty,Ty)\le kG(x,y,y) \quad (1.2)$$

Where $k\in[0,1]$ . Then $T$ has a unique fixed point.In [2] we showed that a mapping satisfies the condition (1.1) will satisfy condition (1.2) when $k\in\left[\frac{1}{2},1\right)$, we showed in a counter example that condition (1.2) need not imply condition (1.1) (for detail see [2]).

**Definition 7**. Let $(X,G)$ be a G-metric space and $T$ be self mapping on $X$ . Then $T$ is called expansive mapping if there exists a constant $a>1$ such that for all $x,y,z\in X$ , we have

$$G(Tx,Ty,Tz)\ge aG(x,y,z)$$

**Main Results 2:** we start our work by proving the following theorem:

**Theorem 2.1**: Let $(X,G)$ be a complete G-metric space and let $T:X\to X$ be an onto mappings satisfies the following condition for all $x,y,z\in X$

$$\begin{aligned}G(Tx,Ty,Tz)&\ge a_1G(x,y,z)+a_2G(x,Tx,Tx)+a_3G(y,Ty,Ty)\\&+a_4G(x,Tz,Tz)+a_5\left[\frac{G(x,Ty,Ty)+G(y,Tx,Tx)}{2}\right]\\&+a_6\left[\frac{G(y,Tz,Tz)+G(z,Ty,Ty)}{2}\right]+a_7\left[\frac{G(x,Tz,Tz)+G(z,Tx,Tx)}{2}\right]\end{aligned} \quad (2.1$$

where $a_1+a_2+a_3+a_4+a_5+a_6+a_7>1$ and $2a_3+2a_4+a_5+2a_6+a_7>2$ . Then $T$ has a unique fixed point.

**Proof**: Let $x_0\in X$ , since $T$ is onto there exist $x_1\in T^{-1}(x_0)$. Continuing in this way, we get a sequence $\{x_n\}$ ,where $x_n\in T^{-1}(x_{n-1})$.If $x_n=x_{n-1}$ for some $n$, we get $x_n$ as a fixed point of $T$. Assuming $x_n\ne x_{n-1}$, for every $n$ ,then from (2.1)

$$G(x_{n-1},x_{n-1},x_n)=G(Tx_n,Tx_n,Tx_{n+1})$$
$$\geq a_1G(x_n,x_n,x_{n+1})+a_2G(x_n,Tx_n,Tx_n)$$
$$+a_3G(x_n,Tx_n,Tx_n)+a_4G(x_{n+1},Tx_{n+1},Tx_{n+1})$$
$$+a_5\left[\frac{G(x_n,Tx_n,Tx_n)+G(x_n,Tx_n,Tx_n)}{2}\right]$$
$$+a_6\left[\frac{G(x_n,Tx_{n+1},Tx_{n+1})+G(x_{n+1},Tx_n,Tx_n)}{2}\right]$$
$$+a_7\left[\frac{G(x_n,Tx_{n+1},Tx_{n+1})+G(x_{n+1},Tx_n,Tx_n)}{2}\right]$$

$$\geq a_1G(x_n,x_n,x_{n+1})+a_2G(x_n,x_{n-1},x_{n-1})$$
$$+a_3G(x_n,x_{n-1},x_{n-1})+a_4G(x_{n+1},x_n,x_n)$$
$$+a_5\left[\frac{G(x_n,x_{n-1},x_{n-1})+G(x_n,x_{n-1},x_{n-1})}{2}\right]$$
$$+a_6\left[\frac{G(x_n,x_n,x_n)+G(x_{n+1},x_{n-1},x_{n-1})}{2}\right]$$
$$+a_7\left[\frac{G(x_n,x_n,x_n)+G(x_{n+1},x_{n-1},x_{n-1})}{2}\right]$$

$$\geq a_1G(x_n,x_n,x_{n+1})+a_2G(x_n,x_{n-1},x_{n-1})$$
$$+a_3G(x_n,x_{n-1},x_{n-1})+a_4G(x_{n+1},x_n,x_n)$$
$$+a_5G(x_n,x_{n-1},x_{n-1})+a_6\left[\frac{G(x_{n+1},x_n,x_n)+G(x_n,x_{n-1},x_{n-1})}{2}\right]$$
$$+a_7\left[\frac{G(x_{n+1},x_n,x_n)+G(x_n,x_{n-1},x_{n-1})}{2}\right]$$
$$\geq\left(a_1+a_4+\frac{a_6}{2}+\frac{a_7}{2}\right)G(x_n,x_n,x_{n+1})$$
$$+\left(a_2+a_3+a_5+\frac{a_6}{2}+\frac{a_7}{2}\right)G(x_n,x_{n-1},x_{n-1})$$
$$\Rightarrow 2G(x_{n-1},x_{n-1},x_n)\geq(2a_1+2a_4+a_6+a_7)G(x_n,x_n,x_{n+1})$$
$$+(2a_2+2a_3+2a_5+a_6+a_7)G(x_n,x_{n-1},x_{n-1})$$
$$\Rightarrow(2-2a_2-2a_3-2a_5-a_6-a_7)G(x_n,x_{n-1},x_{n-1})$$
$$\geq(2a_1+2a_4+a_6+a_7)G(x_n,x_n,x_{n+1})$$
$$\Rightarrow\quad\leq\left(\frac{2-2a_2-2a_3-2a_5-a_6-a_7}{2a_1+2a_4+a_6+a_7}\right)G(x_n,x_{n-1},x_{n-1})$$
$$\Rightarrow G(x_n,x_n,x_{n+1})\leq kG(x_n,x_{n-1},x_{n-1})$$

Where $k=\left(\frac{2-2a_2-2a_3-2a_5-a_6-a_7}{2a_1+2a_4+a_6+a_7}\right)$

Proceeding in this way, we get

$$G(x_n,x_{n+1},x_{n+1})\leq k^nG(x_0,x_1,x_1)$$

Then for all $n,m\in N,m>n$, we have

$$G(x_m,x_n,x_n)\leq G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2})$$
$$+\ldots\ldots\ldots\ldots\ldots+G(x_{m-1},x_m,x_m)$$
$$\leq\left[k^n+k^{n+1}+k^{n+2}+\ldots\ldots\ldots+k^{m-1}\right]G(x_0,x_1,x_1)$$
$$\leq\frac{k^n}{1-k}G(x_0,x_1,x_1)$$

So, $\lim_{n\to\infty}G(x_m,x_n,x_n)\to 0$ as $n,m\to\infty$ and $<x_n>$ is G-Cauchy sequence. By the completeness of $(X,G)$ there exists $u\in X$ such that $<x_n>$ is G-converges to $u$ .

Let $y\in T^{-1}(u)$. For infinitely many $n$ , $x_n\neq u$ . For such $n$ , we have

$$G(x_n,u,u)=G(Tx_{n+1},Ty,Ty)$$

$$\geq a_1G(x_{n+1},y,y)+a_2G(x_{n+1},x_n,x_n)+a_3G(y,Ty,Ty)$$
$$+a_4G(x_{n+1},Ty,Ty)+a_5\left[\frac{G(x_{n+1},Ty,Ty)+G(y,x_n,x_n)}{2}\right]$$
$$+a_6\left[\frac{G(x_{n+1},Ty,Ty)+G(y,Ty,Ty)}{2}\right]$$
$$+a_7\left[\frac{G(x_{n+1},Ty,Ty)+G(x_{n+1},x_n,x_n)}{2}\right]$$

$$\geq a_1G(x_{n+1},y,y)+a_2G(x_{n+1},x_n,x_n)+a_3G(y,Ty,Ty)$$
$$+a_4G(x_{n+1},Ty,Ty)+a_5\left[\frac{G(x_{n+1},Ty,Ty)+G(y,x_n,x_n)}{2}\right]$$
$$+a_6\left[\frac{G(x_{n+1},Ty,Ty)+G(y,Ty,Ty)}{2}\right]$$
$$+a_7\left[\frac{G(x_{n+1},Ty,Ty)+G(x_{n+1},x_n,x_n)}{2}\right]$$

Taking limit $n\to\infty$ we get

$$G(y,u,u) \geq \left(a_3 + a_4 + \frac{a_5}{2} + a_6 + \frac{a_7}{2}\right) G(u,u,y)$$

$$\Rightarrow 2G(y,u,u) \geq (2a_3 + 2a_4 + a_5 + 2a_6 + a_7) G(u,u,y)$$

Which is contradiction since $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 > 1$. So $u = y$, but $Ty = u = y$. Therefore, $u$ is a fixed point of $T$. Suppose there is another fixed point $v$ such that $Tv = v$, then

$$G(u,u,v) = G(Tu,Tu,Tv)$$

$$\geq a_1 G(u,u,v) + a_2 G(u,Tu,Tu) + a_3 G(u,Tu,Tu)$$
$$+ a_4 G(v,Tv,Tv) + a_5 \left[\frac{G(u,Tu,Tu) + G(u,Tu,Tu)}{2}\right]$$
$$+ a_6 \left[\frac{G(u,Tv,Tv) + G(v,Tu,Tu)}{2}\right]$$
$$+ a_7 \left[\frac{G(u,Tv,Tv) + G(v,Tu,Tu)}{2}\right]$$
$$\Rightarrow G(u,u,v) \geq \left(a_1 + \frac{a_6}{2} + \frac{a_7}{2}\right) G(u,u,v) + \left(\frac{a_6}{2} + \frac{a_7}{2}\right) G(u,v,v)$$
$$\Rightarrow 2G(u,v,v) \geq (2a_1 + a_6 + a_7) G(u,u,v) + (a_6 + a_7) G(u,v,v)$$
$$\Rightarrow (2 - 2a_1 - a_6 - a_7) G(u,u,v) \geq (a_6 + a_7) G(u,v,v)$$

Implies that

$G(u,u,v) = 0$. So $u = v$

Hence $T$ has a unique fixed point.

**Theorem 3**: Let $(X,G)$ be a complete G-metric space and $f,g$ be two self mappings on $(X,G)$ satisfying the following conditions:

1. $f(X) \subseteq g(X)$ (3.1)
2. $f$ or $g$ is continuous (3.2)
3. $$G(fx, fy, fz) \geq \alpha \frac{G(fx, gy, gz)}{1 + G(gx, fy, gz)} + \beta \frac{G(gx, fy, gz)}{1 + G(fx, gy, gz)} + \gamma \frac{G(gx, gy, fz)}{1 + G(fx, gy, gz) G(gx, fy, gz)} \quad (3.3)$$
   For every $x, y, z \in X$ and $\alpha, \beta, \gamma \geq 0$ with $3\beta + 3\gamma > 1$. Then $f$ and $g$ have a unique common fixed point in $X$. Provided $f$ and $g$ are weakly compatible maps.

**Proof:** Let $x_0$ be an arbitrary point in $X$. By (3.3), one can choose a point $x_1 \in X$ such that $fx_0 = gx_1$. In general one can choose $x_{n+1}$ such that $y_n = fx_n = gx_{n+1}$, $n = 0,1,2,3........$ From (3.3), we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \geq \alpha \frac{G(fx_n, gx_{n+1}, gx_{n+1})}{1 + G(gx_n, fx_{n+1}, gx_{n+1})}$$
$$+ \beta \frac{G(gx_n, fx_{n+1}, gx_{n+1})}{1 + G(fx_n, gx_{n+1}, gx_{n+1})}$$
$$+ \gamma \frac{G(gx_n, gx_{n+1}, fx_{n+1})}{1 + G(fx_n, gx_{n+1}, gx_{n+1}) G(gx_n, fx_{n+1}, gx_{n+1})}$$
$$\geq \alpha \frac{G(fx_n, fx_n, fx_n)}{1 + G(fx_{n-1}, fx_{n+1}, fx_n)} + \beta \frac{G(fx_{n+1}, fx_{n+1}, fx_n)}{1 + G(fx_n, fx_n, fx_n)}$$
$$+ \gamma \frac{G(fx_{n-1}, fx_n, fx_{n+1})}{1 + G(fx_n, fx_n, fx_n) G(fx_{n-1}, fx_{n+1}, fx_n)}$$
$$\Rightarrow G(fx_n, fx_{n+1}, fx_{n+1}) \geq (\beta + \gamma) G(fx_{n-1}, fx_n, fx_{n+1}) \quad (3.4)$$

By the rectangle inequality of G-metric space we have

$$G(fx_{n-1}, fx_n, fx_{n+1}) \leq G(fx_{n-1}, fx_n, fx_n) + G(fx_{n-1}, fx_n, fx_{n+1})$$
$$= G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1})$$

[By using proposition (1.6)]

From (3.4)

$$G(fx_n, fx_{n+1}, fx_{n+1}) \geq (\beta + \gamma) G(fx_{n-1}, fx_n, fx_n)$$
$$+ (2\beta + 2\gamma) G(fx_n, fx_n, fx_n)$$
$$\Rightarrow (1 - 2\beta - 2\gamma) G(fx_n, fx_{n+1}, fx_{n+1}) \geq (\beta + \gamma) G(fx_{n-1}, fx_n, fx_n)$$
$$\Rightarrow G(fx_{n-1}, fx_n, fx_n) \leq \left[\frac{1 - (2\beta + 2\gamma)}{(\beta + \gamma)}\right] G(fx_n, fx_{n+1}, fx_{n+1})$$
$$\Rightarrow G(fx_{n-1}, fx_n, fx_n) \leq k G(fx_n, fx_{n+1}, fx_{n+1})$$

Where $k = \dfrac{1 - (2\beta + 2\gamma)}{\beta + \gamma} < 1$ as $(3\beta + 3\gamma > 1)$

Continue in the same way we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq k^n G(fx_0, fx_1, fx_1)$$

Therefore, for all $n, m \in N, n < m$, we have by rectangle inequality that

$$G(y_n,y_m,y_m)\le G(y_n,y_{n+1},y_{n+1})+G(y_{n+1},y_{n+2},y_{n+2})$$
$$+\dots\dots\dots+G(y_{m-1},y_m,y_m)$$
$$\le (k^n+k^{n+1}+k^{n+2}+\dots+k^{m-1})G(y_0,y_1,y_1)$$
$$\le \left(\frac{k^n}{1-k}\right)G(y_0,y_1,y_1)$$

Letting $n,m\to\infty$, we have $\lim_{n\to\infty} G(y_n,y_m,y_m)=0.$ Thus $\{y_n\}$ is a G-Cauchy sequence in $X$ . Since $(X,G)$ is complete G-metric space, therefore there exists a point $z\in X$ such that $\lim_{n\to\infty} y_n=\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_{n+1}=z.$ Since the mappings $f$ or $g$ is continuous, therefore $\lim_{n\to\infty} gfx_n=\lim_{n\to\infty} ggx_n=gz$ .Further $f$ and $g$ are compatible, therefore $\lim_{n\to\infty} G(fgx_n,gfx_n,gfx_n)=0.$ Implies $\lim_{n\to\infty} fgx_n=gz$ , from (3.3) we have

$$G(fgx_n,fx_n,fx_n)\ge \alpha\frac{G(fgx_n,gx_n,gx_n)}{1+G(gx_n,fx_n,gx_n)}$$
$$+\beta\frac{G(ggx_n,fx_n,gx_n)}{1+G(fgx_n,gx_n,gx_n)}$$
$$+\gamma\frac{G(ggx_n,gx_n,fx_n)}{1+G(fgx_n,gx_n,gx_n)G(ggx_n,fx_n,gx_n)}$$

$$\Rightarrow G(gz,z,z)\ge \alpha\frac{G(gz,z,z)}{1+G(z,z,z)}+\beta\frac{G(gz,z,z)}{1+G(gz,z,z)}$$
$$+\gamma\frac{G(gz,z,z)}{1+G(gz,z,z)G(gz,z,z)}$$

Proceeding limit as $n\to\infty$, we have $gz=z$ .This implies $z$ is fixed point of $g$ .Again from (3.3)

$$G(fx_n,fz,fz)\ge \alpha\frac{G(fx_n,gz,gz)}{1+G(gx_n,fz,gz)}+\beta\frac{G(gx_n,fz,gz)}{1+G(fx_n,gz,gz)}$$
$$+\gamma\frac{G(gx_n,gz,fz)}{1+G(fx_n,gz,gz)G(gx_n,fz,gz)}$$
$$\ge \alpha\frac{G(z,z,z)}{1+G(z,fz,z)}+\beta\frac{G(z,fz,z)}{1+G(z,z,z)}+\gamma\frac{G(z,z,fz)}{1+G(z,z,z)G(z,fz,z)}$$

$\Rightarrow G(z,fz,fz)\ge(\beta+\gamma)G(z,fz,z)$ Taking limit $n\to\infty$ we have $fz=z$ .This implies $z$ is fixed point of $f$.

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